Meeting 6 Math 22

Section 10.1 Limits

Reading Debrief

- Discuss 4-7 from Reading 5 w/ person next to you.
- Questions about anything from Section 10.1?

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$. The graph: GeoGebra: Reading 5 Task 4(b).

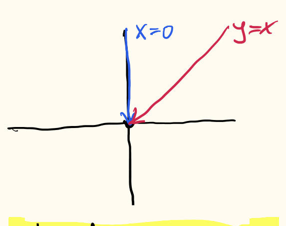
$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{}+1}{\sqrt{}+1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2+1 \quad -1} = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1$$

$$= 2$$

Removable Discontinuity

(b) $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{3x^2}{3x^2+5y^2}$. The graph: GeoGebra: Reading 5 Task 5(b)



$r(t) = \langle 0, t\rangle, \quad t \geq 0$

$$\lim_{t\to 0} \frac{3x(t)^2}{3x(t)^2+5y(t)^2} = \lim_{t=0} 0 = 0$$

$r(t) = \langle t, t\rangle, \quad t \geq 0$

$$\lim_{t\to 0} \frac{3t^2}{3t^2+5t^2} = \lim_{t\to 0} \frac{3}{8} = \frac{3}{8}$$

The limit does not exist because the limits along two different paths do not agree.

Section 10.2 First-Order Partial Derivatives

Let $f(x,y)$ be a function and (a,b) a point. We can get a single variable function

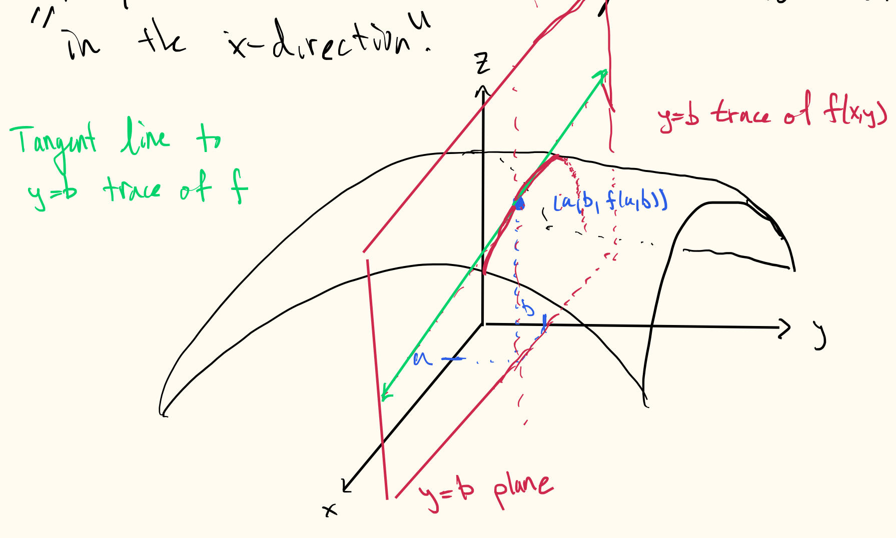
$$g(x) = f(x,b)$$

by setting $y=b$. the graph of $g(x)$ is the $y=b$ trace of $f(x,y)$. Since $g(x)$ is a single variable function, we can define its derivative w.r.t. x

$$g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}.$$

this limit is the slope of the tangent line to the $y=b$ trace of $f(x,y)$ at the point $x=a$. Thus, it measures the rate of change of $f(x,y)$ at (a,b) "in the x-direction".



the derivative $g'(a)$ is called the partial derivative of f with respect to x.

Definition 10.2.4 The first-order partial derivatives of f with respect to x and y at a point (a,b) are defined via

- $\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$
- $\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$

when these limits exist.

Activity 10.2.2

- Complete Activity 10.2.2 and discuss w/ your group.
- Class discussion

Activity 10.2.2. Consider the function f defined by

$$f(x,y) = \frac{xy^2}{x+1}$$

at the point $(1,2)$.

a. Write the trace $f(x,2)$ at the fixed value $y=2$. On the left side of Figure 10.2.5, draw the graph of the trace with $y=2$ around the point where $x=1$, indicating the scale and labels on the axes. Also, sketch the tangent line at the point $x=1$.

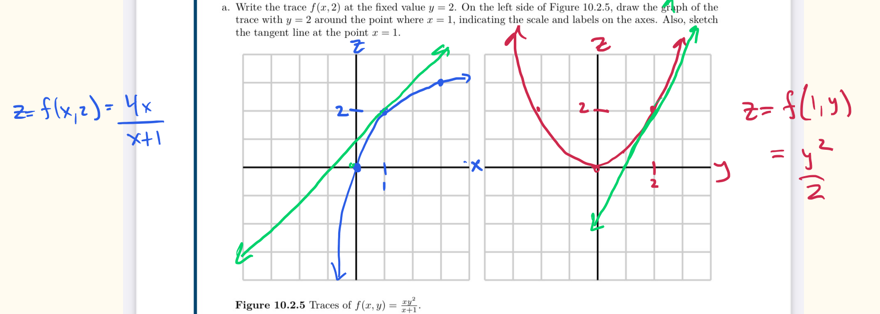


Figure 10.2.5 Traces of $f(x,y) = \frac{xy^2}{x+1}$.

b. Find the partial derivative $f_x(1,2)$ and relate its value to the sketch you just made.

c. Write the trace $f(1,y)$ at the fixed value $x=1$. On the right side of Figure 10.2.5, draw the graph of the trace with $x=1$ indicating the scale and labels on the axes. Also, sketch the tangent line at the point $y=2$.

d. Find the partial derivative $f_y(1,2)$ and relate its value to the sketch you just made.

(b) $f_x(x,y) = y^2\frac{\partial}{\partial x}\left(\frac{x}{x+1}\right) = y^2\left(\frac{x+1-x}{(x+1)^2}\right)$

$= y^2/(x+1)^2$

$f_x(1,2) = 1$ = slope of tangent line to blue graph at $x=1$

(d) $f_y(x,y) = \frac{2xy}{x+1} \qquad f_y(1,2) = 2$

Activity 10.2.3

- Complete together as a class.

Activity 10.2.3.

a. If $f(x,y) = 3x^3 - 2x^2y^5$, find the partial derivatives f_x and f_y.

b. If $f(x,y) = \frac{xy^2}{x+1}$, find the partial derivatives f_x and f_y.

c. If $g(r,s) = rs\cos(r)$, find the partial derivatives g_r and g_s.

d. Assuming $f(w,x,y) = (6w+1)\cos(3x^2+4xy^3+y)$, find the partial derivatives f_w, f_x, and f_y.

e. Find all possible first-order partial derivatives of $q(x,t,z) = \frac{x2^tz^3}{1+x^2}$.

(a) $f_x(x,y) = 9x^2 - 4xy^5$

$f_y(x,y) = -10x^2y^4$

(c) $g_r(r,s) = s\cos(r) - rs\sin(r)$

$g_s(r,s) = r\cos(r)$

(d) $f_w = 6\cos(3x^2+4xy^3+y)$

$f_x = -(6w+1)\sin(3x^2+4xy^3+y)\cdot(6x+4y^3)$

$f_y = -(6w+1)\sin(3x^2+4xy^3+y)\cdot(12xy^2+1)$

(e) Skip.

Section 10.2.2 Interpretations of First-Order Partials

Activity 10.2.4

- Complete w/ your group
- Class discussion.

Activity 10.2.4. The speed of sound C traveling through ocean water is a function of temperature, salinity and depth. It may be modeled by the function

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S-35) + 0.016D.$$

Here C is the speed of sound in meters/second, T is the temperature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

a. State the units in which each of the partial derivatives, C_T, C_S and C_D, are expressed and explain the physical meaning of each.

b. Find the partial derivatives C_T, C_S and C_D.

c. Evaluate each of the three partial derivatives at the point where $T=10$, $S=35$ and $D=100$. What does the sign of each partial derivatives tell us about the behavior of the function C at the point $(10,35,100)$?

$C(T,S,D)$

(a)
- $C_T = \lim_{h\to 0} \frac{C(T+h,S,D)-C(T,S,D)}{h} = \frac{m/s}{°C}$

  C_T is the instantaneous rate of change of C with respect to temp at a fixed salinity and depth.
- $C_S = \frac{m/s}{g/L} = \frac{m\cdot L}{g\cdot s}$

  C_S is the instantaneous rate of change of C with respect to salinity at a fixed temp. and depth.
- $C_D = \frac{m/s}{m} = \frac{1}{s} = hz$

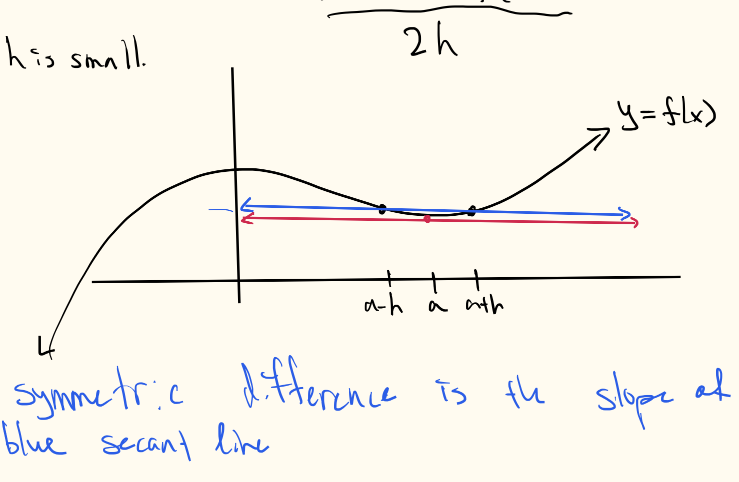
  C_D is the instantaneous rate of change of C with respect to depth at a fixed temp. and salinity.

Section 10.2.3 Estimating Partial Derivatives

The derivative $f'(a)$ of a single-variable function can be estimated using the "symmetric difference"

$$f'(a) \approx \frac{f(a+h)-f(a-h)}{2h}$$

when h is small.



The symmetric difference is the slope of the blue secant line

Activity 10.2.5

- Complete w/ your group.
- Class discussion.

$w(v,T)$ = wind chill

v\T	−30	−25	−20	−15	−10	−5	0	5	10	15	20
5	−46	−40	−34	−28	−22	−16	−11	−5	1	7	13
10	−53	−47	−41	−35	−28	−22	−16	−10	−4	3	9
15	−58	−51	−45	−39	−32	−26	−19	−13	−7	0	6
20	−61	−55	−48	−42	−35	−29	−22	−15	−9	−2	4
25	−64	−58	−51	−44	−37	−31	−24	−17	−11	−4	3
30	−67	−60	−53	−46	−39	−33	−26	−19	−12	−5	1
35	−69	−62	−55	−48	−41	−34	−27	−21	−14	−7	0
40	−71	−64	−57	−50	−43	−36	−29	−22	−15	−8	−1

a. Estimate the partial derivative $w_v(20,-10)$. What are the units on this quantity and what does it mean? (Recall that we can estimate a partial derivative of a single variable function f using the symmetric difference quotient $\frac{f(x+h)-f(x-h)}{2h}$ for small values of h. A partial derivative is a derivative of an appropriate trace.)

b. Estimate the partial derivative $w_T(20,-10)$. What are the units on this quantity and what does it mean?

c. Use your results to estimate the wind chill $w(18,-10)$. (Recall from single variable calculus that for a function f of x, $f(x+h) \approx f(x) + hf'(x)$.)

d. Use your results to estimate the wind chill $w(20,-12)$.

e. Consider how you might combine your previous results to estimate the wind chill $w(18,-12)$. Explain your process.

Units: w = °F

v = mi/h

T = °F

(a) $w_v(20,-10) \approx \frac{w(20+h,-10)-w(20-h,-10)}{2\cdot h}$

$\approx \frac{w(25,-10)-w(15,-10)}{10} = \frac{-37-(-32)}{10}$

$= -\frac{1}{2}$ °F/mi/h

When temp is -10, you will feel $-\frac{1}{2}$°F colder for every 1 mi/h increase in wind speed.

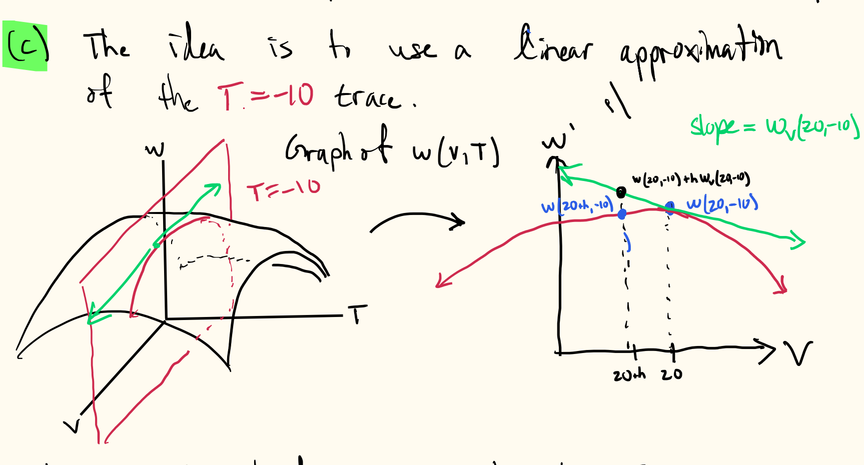
(b) $w_T(20,-10) \approx \frac{w(20,-10+5)-w(20,-10-5)}{10}$

$= \frac{-29-(-42)}{10} = 1.3$ °F/°F

no units?

When the wind is 20mph you will feel 1.3°F warmer per 1°F increase in ambient temp.

(c) The idea is to use a linear approximation of the $T=-10$ trace.



The green tangent line approximates the $T=-10$ trace near $v=20$. For small values of h, we have

linear approx

$$w(20+h,-10) \approx w(20,-10) + h\,w_v(20,-10)$$

Using $h=-2$,

$$w(18,-10) \approx w(20,-10) - 2w_v(20,-10)$$

$$= -35 - 2\cdot\left(-\tfrac{1}{2}\right) = -34°F$$

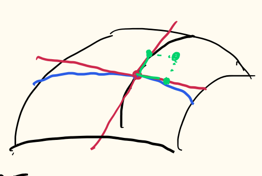
(d) Similar to (c), using $v=20$ trace.

$$w(20,-12) \approx w(20,-10) + (-2)w_T(20,-10)$$

$$= -35 - 2.6 = -37.6°F$$

(e) $w(18,-12) = w(20,-10) + (-2)w_T(20,-10) + (-2)w_v(20,-10)$

$= -35 - 2.6 + 1 = -36.6°F$



Approximate w/ tangent plane!